On Mason's rigidity theorem

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October 4, 2008

Abstract

Following an argument proposed by Mason, we prove that there are no algebraically special asymptotically simple vacuum space-times with a smooth, shear-free, geodesic congruence of principal null directions extending transversally to a cross-section of $\mathscr{I}^+$. Our analysis leaves the door open for escaping this conclusion if the congruence is not smooth, or not transverse to $\mathscr{I}^+$. One of the elements of the proof is a new rigidity theorem for the Trautman-Bondi mass.

Contents

1 Introduction

It is a long standing conjecture that the only vacuum algebraically special asymptotically simple space-time is Minkowski space. Arguments towards a

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proof have been presented in [14]. The aim of this work is to establish the conjecture under a set of restrictive conditions, with the aid of a rigidity theorem for Trautman-Bondi mass, a complete proof of which has not been presented previously.

A space-time $(\mathscr{M}, g)$ is said to admit a conformal boundary completion at infinity if there exists a manifold $\widetilde{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}$ with boundary $\mathscr{I}$ and a function Ω on $\widetilde{\mathscr{M}}$ vanishing precisely on $\mathscr{I}$, with nowhere vanishing gradient there, such that the metric $\widetilde{g} := \Omega^2 g$ extends smoothly to a tensor field with Lorentzian signature defined on $\widetilde{\mathscr{M}}$. We denote by $\mathscr{I}^+$, respectively $\mathscr{I}^-$, the set of points on $\mathscr{I}$ which are end-points of null future directed, respectively past directed, geodesics. We will say that[1] $(\mathscr{M}, g)$ is *past asymptotically simple* if every maximally extended null geodesic acquires a past end-point on $\mathscr{I}^- \subset \mathscr{I}$. *Future asymptotic simplicity* is defined by changing time-orientation. Following [17], we use the term *asymptotic simplicity* if $\mathscr{M}$ does not contain closed timelike curves, and if both past and future asymptotic simplicity hold. An embedded submanifold of $\mathscr{I}$ will be said to be a *cross-section* if it meets generators of $\mathscr{I}$ transversally, and at most once each.

Asymptotically simple space-times with null conformal boundaries are known to be globally hyperbolic [17], with contractible Cauchy surfaces, with $\mathscr{I}^+$ and $\mathscr{I}^-$ containing $\mathbb{R} \times S^2$, where the $\mathbb{R}$ factor corresponds to motions along the null geodesic generators. Furthermore, $\mathscr{I}$ reduces to two copies of $\mathbb{R} \times S^2$ if one assumes that the extended space-time $(\widetilde{\mathscr{M}}, \widetilde{g})$ is strongly causal at $\mathscr{I}$.

It appears of some interest to consider algebraically special space-times which are asymptotically simple to the past, without necessarily being asymptotically simple.[2] Such space-times could describe e.g. the formation of a black hole in a space-time without singularities in the past.

Recall that a space-time $(\mathscr{M}, g)$ is *algebraically special* if at every point there exists a null vector ℓ such that the Weyl tensor $C_{\mu\nu\rho\sigma}$ satisfies

$$C_{\mu\nu\rho[\sigma}\ell_{\pi]}\ell^\rho\ell^\nu = 0 \,. \tag{1.1}$$

Assume that $(\mathscr{M}, g)$ is vacuum, or that the Ricci tensor satisfies a set of restrictions listed in detail below. Then, on regions where the ℓ's can be chosen to produce a smooth vector field, near those orbits along which the Weyl tensor isn't zero everywhere, ℓ can be rescaled[3] so that its integral curves are null affinely parameterized geodesics without shear. Conversely, the existence of such a congruence implies (1.1).

In asymptotically simple space-times the integral curves of ℓ extend smoothly to the conformal boundary at their end points. The question then arises, whether a suitable rescaling $\tilde{\ell}$ of ℓ extends by continuity to a smooth vector

[1] Recall that a space-time is a time-oriented Lorentzian manifold, the topology of which is assumed to be metrisable. In view of our extensive use of the NP formalism the signature $(+---)$ is used.

[2] We are grateful to an anonymous referee for pointing out this possibility to us.

[3] A priori this can be done only locally; however, in globally hyperbolic space-times (which is the case here) this can always be done globally when ℓ is globally smooth.

field defined on the set

$$\mathscr{M} \cup \underbrace{\{p \in \mathscr{I} \,|\, p \text{ is an end point of precisely one integral curve of } \ell\}}_{\mathscr{U}}$$
$$\subset \widetilde{\mathscr{M}} := \mathscr{M} \cup \mathscr{I} \; .$$

Since the integral curves of ℓ intersect $\mathscr{I}$ transversally, $\tilde{\ell}$ is transverse to $\mathscr{U}$. However, neither continuity nor differentiability of $\tilde{\ell}$ at $\mathscr{U}$ are clear. Further, $\tilde{\ell}$ might become singular as the boundary $\overline{\mathscr{U}} \setminus \mathscr{U}$ is approached, or perhaps develop zeros there. Problems will clearly arise at points at which more than one integral curve of ℓ meets $\mathscr{I}^+$, assuming that $\tilde{\ell}$ can be defined at those points at all. One of our results here (see Appendix B) is the proof that smoothness and transversality to $\mathscr{I}^+$ of

$$\tilde{\ell} := \Omega^{-2}\ell$$

is equivalent to the non-existence of zeros of the complex divergence $\rho = \bar{m}^\mu m^\nu \nabla_\mu \ell_\nu$ (see, e.g., [16] for details of the definition of m^μ) of the congruence defined by ℓ in a neighborhood of $\mathscr{I}^+$.

Assuming that ℓ is globally well defined, in Section 4 we prove:

Theorem 1.1 *The following set of conditions is incompatible in vacuum:*

1. *$(\mathscr{M}, g)$ is past asymptotically simple and contains a contractible Cauchy surface.*[4]

2. *There exists on $\mathscr{M}$ a smooth, null, shear-free, geodesic vector field ℓ.*

3. *We have $\mathscr{I}^- \approx \mathbb{R} \times S^2$, and there exists a compact cross-section S^+ of $\mathscr{I}^+$ near which a rescaling $\tilde{\ell}$ of ℓ extends by continuity to a smooth vector field which is transverse to $\mathscr{I}^+$.*

The statement remains true for non-vacuum metrics if the dominant energy condition holds and if the Newman-Penrose components Φ_{00}, Φ_{01}, Φ_{02} and Λ of the Ricci tensor[5]*, associated to the congruence defined by ℓ, vanish, with the remaining components decaying fast enough.*[6]

Remark 1.2 Theorem 1.1 holds under finite differentiability conditions on $\tilde{\ell}$, but we have not attempted to determine the threshold; in any case there are no *a priori* reasons for a geodesic shear-free null congruence to be globally C^0 or C^1, even if the metric is smooth. Indeed, poorly differentiable examples can be constructed in Minkowski space-time, see Section 2 where somewhat more general congruences are allowed.

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[4] As already pointed out, all these conditions will hold by [17] if the space-time is asymptotically simple.

[5] Here and elsewhere, we follow the conventions of [16] for the NP spin-coefficient formalism, so that these conditions on the Ricci tensor are equivalent to the vanishing of the scalar curvature and of $\Phi_{ABA'B'}o^A o^B$, where o^A is the spinor obtained from ℓ.

[6] The exact decay rates needed can be found by chasing through the calculations in [15, 25] that lead to the Natorf-Tafel mass aspect formula (3.33) below.

Remark 1.3 As can be seen from footnote 5, the Ricci tensor conditions of Theorem 1.1 will hold in electro-vacuum if $o^A o^B \varphi_{AB} = 0$, where φ is the Maxwell spinor, with o^A as in Section 3.

Theorem 1.1 is similar in spirit to the results of Mason [14]. The differences between our hypotheses and those of [14] are as follows: First, algebraic speciality does *not* imply the *smoothness* of either $\tilde{\ell}$ or ℓ. Next, neither existence nor transversality of $\tilde{\ell}$ at $\mathscr{I}^+$ are assumed in [14]. We further note that the hypotheses of Theorem 1.1 enforce non-vanishing of twist throughout a region of $\mathscr{M}$ relevant for the proof (see Proposition 3.1 below), while more general configurations are *a priori* allowed in [14].[7] Finally, our argument requires the topology of $\mathscr{I}^-$ to be $\mathbb{R} \times S^2$ which, for asymptotically simple space-times as considered in [14], is only known to be true [17] when $\mathscr{M} \cup \mathscr{I}^-$ is strongly causal.

The key idea stems from [14], but some steps of the argument require careful reorganizations. The proof can be structured as follows: We start, in Section 3, by introducing a coordinate system based on the members of the congruence. This allows us to construct a cut S^- of $\mathscr{I}^-$, associated to the cut S^+ of $\mathscr{I}^+$, on which $\tilde{\ell}$ is transverse. The calculations in [14] subsequently show that the Trautman-Bondi mass $m_{\mathrm{TB}}(S^+)$ of S^+ is the negative of that of S^-.

One then wishes to appeal to the positive energy theorem to show flatness of the metric near a cross-section of $\mathscr{I}^+$. This requires controlled spacelike hypersurfaces, say $\mathscr{S}$, which are constructed at the beginning of Section 4. So, the positive energy theorem of [7] implies that $m_{\mathrm{TB}}(S^+)$ vanishes, and that $\mathscr{S}$ carries a timelike KID. An analysis of Killing developments allows one to conclude that the initial data on the $\mathscr{S}$ can be realized by embedding in Minkowski space-time; this is in fact a new rigidity result for the Trautman-Bondi mass, see Theorem 4.1. One concludes by showing that no congruences with the properties listed exist near a Minkowskian $\mathscr{I}^+$.

We shall say that an algebraically special space-time is *non-branching* if it is either type II or D everywhere[8], or type III everywhere, or type N everywhere. The point is that in these cases the Weyl tensor does not allow branching of the principal null directions. We then have the following related statement:

Theorem 1.4 *The following conditions are incompatible:*

1. *$(\mathscr{M}, g)$ is past asymptotically simple and contains a contractible Cauchy surface.*[4]

2. *$(\mathscr{M}, g)$ is non-branching, vacuum, with $\mathscr{I}^- \approx \mathbb{R} \times S^2$, and the complex divergence ρ of the congruence has no zeros near a compact cross-section S^+ of $\mathscr{I}^+$.*

The conclusion remains true for non-vacuum space-times if the conditions on the Ricci tensor spelled out in Theorem 1.1 are met.

[7]Theorem 2.1 below allows configurations somewhat more general than Theorem 1.1, but those are still less general than indicated in [14].

[8]We allow the metric to be II at some places and D at others.

Indeed, assume that such a space-time exists. We show in Appendix A that ℓ can be chosen to be smooth throughout $\mathscr{M}$, and in Appendix B that $\Omega^{-2}\ell$ is smooth and transverse at S. By Proposition 4.5 below $(\mathscr{M}, g)$ contains a flat region, thus is of type O there, which gives a contradiction.

2 Non-differentiable congruences

It appears of interest to find a set of hypotheses, alternative to those of Theorem 1.1, which are compatible with at least one space-time. A possible direction of enquiries is to admit smoothness and transversality of $\tilde{\ell}$ near one or more sections of $\mathscr{I}^+$, but allow singularities of $\tilde{\ell}$ in the space-time. (The question of non-transversal congruences will be discussed in Section 5.) Now, consider any maximally extended null geodesic γ initially tangent to $\tilde{\ell}$ near $\mathscr{I}^+$. In the argument below it is *necessary* that the tangent to γ remains proportional to $\tilde{\ell}$. If $\tilde{\ell}$ is allowed to become singular, this last property might not hold, and it is easy to construct congruences where this occurs. (Consider, for example, any timelike curve Γ in Minkowski space-time extending from i^- to i^+, let u denote the retarded time function based on Γ, and let $\ell = du$ on $\mathscr{M} \setminus \Gamma$. Then every integral curve of ℓ, when followed from $\mathscr{I}^+$ towards space-time, stops at Γ.) Clearly, any argument in which null geodesics need to be followed from $\mathscr{I}^+$ to $\mathscr{I}^-$ has no chance of succeeding in such situations.

In this last example, of a congruence based on a curve Γ, one can smoothly flow the geodesics through Γ, landing on a second congruence generated by the past light-cones issued from Γ. To accommodate such situations in any kind of generality would require considering multiple-valued congruences.[9]

One could, however, enquire what happens if $\tilde{\ell}$ is smooth on a dense set, and if one further assumes that null geodesics somewhere tangent to $\tilde{\ell}$ remain tangent to $\tilde{\ell}$ at all those points at which $\tilde{\ell}$ is defined. The apparent difficulty of flowing along a singular vector field $\tilde{\ell}$ is easily resolved by flowing along the associated geodesics. Anticipating, in such situations Proposition 3.1 below does not hold anymore, and one faces the problem of understanding what happens along those geodesics on which the twist vanishes. The hypothesis that the space-time is smooth together with the Newman-Penrose equations leads then to the vanishing of some components ψ_i of the Weyl tensor along such geodesics, but the implications of this are not clear. It is conceivable that this might again lead to a mass changing sign as in Proposition 3.5 below, which would allow one to conclude, but this remains to be seen.

In spite of the above, some degree of singularity of ℓ can be allowed, as follows. To obtain more control of the space-time we will assume full asymptotic simplicity, and consider a sequence of spherical cuts S_i^+ of $\mathscr{I}^+$ near which $\tilde{\ell}$ is again assumed to be smooth and transverse. Let us set

$$\begin{aligned} \mathring{S}_i^+ \quad := \quad & \{p \in S_i^+ : \quad \ell \text{ is smooth in an } \mathscr{M}\text{-neighborhood of the maximally} \\ & \text{extended null geodesic with tangent } \tilde{\ell} \text{ at its end point } p\} \, . \end{aligned}$$

[9] The density condition (2.1) below essentially forbids multiple-valued congruences, which therefore appear to be intractable by our arguments.

Rather than assuming that ℓ is smooth throughout $\mathscr{M}$, so that $\mathring{S}_i^+ = S_i^+$, suppose instead that

$$\mathring{S}_i^+ \text{ is dense within } S_i^+ \ . \tag{2.1}$$

Let $\mathscr{I}_0 \subset \mathscr{I}$ be defined as

$$\mathscr{I}_0 := \{p \in \mathscr{I} | \ \text{strong causality holds at } p\} \ ,$$

with $\mathscr{I}_0^{\pm} = \mathscr{I}_0 \cap \mathscr{I}^{\pm}$. According to Newman [17], in asymptotically simple space-times each of $\mathscr{I}_0^{\pm}$ is diffeomorphic to $\mathbb{R} \times S^2$, with the generators of $\mathscr{I}$ tangent to the $\mathbb{R}$ factor, which we parameterize by u; we choose u to be increasing to the future. We let S_i^+ be any S^2 included in $\mathscr{I}_0^+$ that intersects every generator of $\mathscr{I}_0^+$ precisely once; such sets will be called cross-sections of $\mathscr{I}^+$. (It actually follows from Theorem 2.1, which we are about to state, that $\mathscr{I}_0 = \mathscr{I}$ under the hypotheses there.) We claim that:

Theorem 2.1 *Let $(\mathscr{M}, g)$ be an asymptotically simple space-time with smooth null asymptote $(\widetilde{\mathscr{M}}, \widetilde{g})$ such that $\mathscr{I}^- \approx \mathbb{R} \times S^2$. Assume that there exists a sequence of cross-sections S_i^+ of $\mathscr{I}_0^+$, $i \in \mathbb{N}$, such that*

$$\mathscr{I}_0^+ \subset \cup_{i\in\mathbb{N}} J^+(S_i^+, \widetilde{\mathscr{M}})$$

together with a geodesic, shear free, null vector field $\tilde{\ell}$ defined on a neighborhood of

$$\mathscr{M} \cup_{i\in\mathbb{N}} S_i^+$$

satisfying (2.1). Assume that the dominant energy condition holds and that the Newman-Penrose components Φ_{00}, Φ_{01}, Φ_{02} and Λ of the Ricci tensor, associated to the congruence defined by ℓ, vanish, while the remaining ones decay fast enough. If $\tilde{\ell}$ is transverse to $\cup_{i\in\mathbb{N}} S_i^+$, then $(\mathscr{M}, g)$ is the Minkowski space-time $\mathbb{R}^{1,3}$.

Remark 2.2 The Kerr congruence in Minkowski space-time (see, e.g., the appendix to [13]) satisfies the hypotheses of Theorem 2.1.

The proof of Theorem 2.1 can be found at the end of Section 4.

3 The metric form of algebraically-special vacuum solutions

We now run through the derivation of the metric form of algebraically-special space-times, first vacuum and then noting the changes for non-vacuum. References for this are [13, 14, 23]. The general technique is to construct a coordinate system with the aid of the geodesic and shear-free congruence and solve enough of the Newman-Penrose spin-coefficient equations to obtain the radial dependence of the metric. We follow [16] for the Newman-Penrose spin-coefficient equations, rather than the version in [23]. We modify the derivations in [23] and [14] in order to connect the coordinate system to standard coordinates on $\mathscr{I}^+$ from the start of the calculation.

Consider a manifold S^+ transverse to the generators of $\mathscr{I}^+$. We use a local coordinate u along the generators and a local complex coordinate ζ on S^+, so that the (degenerate) metric of $\mathscr{I}^+$ is

$$-4\frac{d\zeta d\overline{\zeta}}{P^2}\ .$$

As the calculations that follow are purely local in u and ζ we can, without loss of generality, choose $P = 1 + \zeta\overline{\zeta}$. (If S^+ is a sphere, then $(u, \zeta, \overline{\zeta})$ are Bondi coordinates at $\mathscr{I}^+$.) Recall that, in $(\widetilde{\mathscr{M}}, \widetilde{g})$, the usual spinor dyad $(\widetilde{O}^A, \widetilde{I}^A)$ and corresponding NP tetrad $(\widetilde{L}^a, \widetilde{N}^a, \widetilde{M}^a, \overline{\widetilde{M}}^a)$ are related to the coordinates by

$$\begin{aligned} \widetilde{N}^a\partial_a &= \partial_u\ , \\ \widetilde{M}^a\partial_a &= \frac{P}{\sqrt{2}}\partial_\zeta\ , \end{aligned}$$

and

$$\begin{aligned} \widetilde{N}_a dx^a &= -d\Omega\ , \\ \widetilde{M}_a dx^a &= -\frac{\sqrt{2}}{P}d\overline{\zeta}\ , \end{aligned}$$

where the last equation is understood as pulled back to $\mathscr{I}^+$, where $d\Omega$ pulls back to zero (the point is that there are different ways of extending the coordinates into the interior).

By assumption, ℓ generates a geodesic and shear-free null congruence and we may scale ℓ so that it is affinely parameterized. Therefore ℓ defines a smooth spinor field o^A, which in turn can be scaled to be parallelly-propagated along the congruence. In the NP formalism, this is

$$Do^A := \ell^b\nabla_b o^A = 0. \tag{3.1}$$

There remains a residual freedom to rescale o^A by a nowhere-zero function F^0 which is constant along the congruence, when ℓ rescales with $|F^0|^2$.

Under conformal-rescaling of the metric $\widetilde{g} = \Omega^2 g$, the rescaling $\tilde{\ell} = \Omega^{-2}\ell$ takes the affinely normalized geodesic vector field ℓ to an affinely normalized geodesic one, leading to a vector field $\tilde{\ell}$ which is continuous at $\mathscr{I}$ by hypothesis. The rescaling $\tilde{o}^A = \Omega^{-1}o^A$ likewise extends to $\mathscr{I}^+$, where

$$\tilde{o}^A = B\widetilde{O}^A + C\widetilde{I}^A \tag{3.2}$$

for smooth functions B and C on $\mathscr{I}^+$. The assumption that $\tilde{\ell}$ is transverse to $\mathscr{I}^+$on S^+ translates to the requirement that B be nonzero on that part of $\mathscr{I}^+$. We extend B and C into the interior as functions constant along the congruence and then rescale $\tilde{o}^A$ to set $B = 1$. Now, at $\mathscr{I}^+$, we have

$$\tilde{o}^A = \widetilde{O}^A + L(u, \zeta, \overline{\zeta})\widetilde{I}^A, \tag{3.3}$$

in terms of a function L on $\mathscr{I}^+$. The assumption of transversality implies that L is a smooth function on S^+. (We could define L independently of the scaling of $\tilde{o}^A$ as $L = \widetilde{O}_A \tilde{o}^A / \widetilde{I}^B \tilde{o}_B$.)

Equation (3.1) implies that the spin-coefficients κ and ϵ are zero and, by assumption, σ is also zero.

We extend the coordinates u and ζ into the interior by taking them to be constant along the geodesics of the congruence, so that

$$Du = 0 = D\zeta \ ,$$

and then $D = \partial/\partial r$ with r as before. This fixes r uniquely up to a shift of origin on each geodesic of the congruence. A convenient (and standard) way to choose the origin in r is next to solve one of the spin-coefficient equations, (A.3a) as given in [16] which, with the restrictions that we currently have on the spin-coefficients, is just:

$$D\rho = \rho^2.$$

The solution of this is either $\rho \equiv 0$ or

$$\rho = -(r + r^0 + i\Sigma)^{-1} \tag{3.4}$$

where r^0 and Σ are real functions of integration, constant along the congruence (and so are functions only of $(u, \zeta, \overline{\zeta})$; as before, we use the superscript 0 for functions independent of r but, as is conventional, omit it from Σ). We choose the origin of r so that $r^0 = 0$. Note that, if $\rho \not\equiv 0$ and Σ ever vanishes, so that the twist of the congruence vanishes, then ρ is real and in this case ρ will diverge at a finite r. This is incompatible with smoothness of the congruence, unless ρ identically vanishes, leading to:

PROPOSITION 3.1 *Under the hypotheses of Theorem 1.1, the divergence ρ and the twist Σ are nowhere vanishing on those integral curves of ℓ which have end points on S^+.*

PROOF: Since $\tilde{\ell}$ is smooth, transverse, and geodesic, we must have $\tilde{\ell} = \chi \Omega^{-2} \ell$ for some smooth nowhere vanishing function χ. We might therefore without loss of generality assume, rescaling $\tilde{\ell}$ if necessary, that $\chi = 1$. From Equations (B.11) and (B.14) of Appendix B we obtain

$$\rho = \Omega^2 \tilde{\rho} + \frac{D\Omega}{\Omega} = -\frac{1}{r} + O(r^{-2}) \ .$$

where $\tilde{\rho}$ is associated with $\tilde{\ell}$ just as ρ is associated with ℓ (see Appendix B for the details of this). We conclude that ρ has no zeros near S^+. Hence ρ is not identically zero on any of the relevant members of the congruence, and since ℓ is smooth by hypothesis, (3.4) excludes zeros of Σ. □

Following Mason [14], who in turn follows Debney et al. [9], we next consider the complex vector field

$$W_a = o^B \nabla_a o_B.$$

By the geodesic, shear-free condition this is of the form $o_A \tau_{A'}$ for some spinor field $\tau_{A'}$ with $\overline{o}^{A'} \tau_{A'} = \rho$. By smoothness of the congruence, ρ is smooth in the interior, and it does not vanish by Proposition 3.1. Therefore we can define the spinor field ι^A, which makes up the NP dyad with o^A, via its complex conjugate by $\overline{\iota}^{A'} = -\rho^{-1} \tau^{A'}$. Then

$$W_a = -\rho o_A \overline{\iota}_{A'} = -\rho m_a, \tag{3.5}$$

and the spin-coefficient τ is also zero.

With the spin-coefficient equations numbered as in [16], from (A.3c) and (A.3p) with what we have now and the vacuum equations we find that π and λ vanish. With the aid of (A.3a), we calculate the exterior derivative of W_a from (3.5) as

$$\nabla_{[a} W_{b]} = X_1 \ell_{[a} m_{b]} + X_2 \overline{m}_{[a} m_{b]}, \tag{3.6}$$

in terms of two functions X_1 and X_2 whose precise form does not concern us. Thus, in the language of differential forms, $W \wedge dW = 0$. We note the following, presumably well known, complex version of the Frobenius theorem:

LEMMA 3.2 *There exist, locally, complex-valued functions X_3 and X_4 such that*

$$W = X_3 dX_4 \; .$$

PROOF: Note that

$$W \wedge \overline{W} = \rho \overline{\rho} m \wedge \overline{m} \neq 0 \tag{3.7}$$

by Proposition 3.1, which shows that the real and the imaginary part of W are nowhere vanishing, linearly independent. Elementary algebra gives $dW = W \wedge Z$ for some complex-valued one-form Z. The usual calculation shows that the two-dimensional distribution defined by the collection of vector fields

$$\{X \in \Gamma TM : \quad W(X) = 0\}$$

is integrable. Hence there exist, locally, complex functions α and β, as well as real valued functions f and g such that

$$W = \alpha df + \beta dg \; .$$

Equation (3.7) shows that α and β are nowhere-vanishing, and that df and dg are linearly independent. The equation $W \wedge dW = 0$ implies $\alpha / \beta = \varphi + i\psi$ for some functions $\varphi = \varphi(f,g)$ and $\psi = \psi(f,g)$, hence

$$W = \beta(\varphi df + dg + i\psi dg) \; .$$

Consider the two-dimensional Riemannian metric

$$b := (\varphi df + dg)^2 + \psi^2 dg^2 \; .$$

By the uniformization theorem there exist, again locally, smooth functions x, y and h such that

$$b = e^{2h} \Big((dx)^2 + (dy)^2 \Big) \; .$$

Changing y to $-y$ if necessary, at each point the b-ON coframes $\{\varphi df + dg, \psi dg\}$ and $\{e^h dx, e^h dy\}$ are rotated with respect to each other, hence there exists a function $\theta = \theta(x, y)$ such that

$$(\varphi df + dg + i\psi dg) = e^{h+i\theta}(dx + idy)\ .$$

The functions $X_3 = \beta e^{h+i\theta}$ and $X_4 = x + iy$ satisfy our claim. □

Returning to the problem at hand, either (3.7), or the argument in the proof of Lemma 3.2, shows that the real and imaginary parts of X_4 are independent functions.

By (3.5) and (3.6), both dX_3 and dX_4 are orthogonal to ℓ, so that both are functions only of $(u, \zeta, \overline{\zeta})$, and we can determine them by looking at the value of W at $\mathscr{I}^+$. We have

$$\begin{aligned} W_a dx^a &= o^B \nabla_a o_B dx^a \\ &= \Omega \tilde{o}^B (\widetilde{\nabla}_{AA'} \tilde{o}_B + \Upsilon_{BA'} \tilde{o}_A) dx^a, \end{aligned}$$

where $\Upsilon_a = \partial_a \log \Omega$, we use the rules for conformal transformation of the spinor connection given in [20] and we use the rescaled dyad of (B.12). We calculate this from (3.3) and pullback to $\mathscr{I}^+$ to find that, at $\mathscr{I}^+$,

$$W_a dx^a = \widetilde{M}_a dx^a = -\frac{\sqrt{2}}{P} d\overline{\zeta}. \tag{3.8}$$

However, from what we have said above about X_3 and X_4, (3.8) holds everywhere, so that, by (3.5), in the interior

$$m_a dx^a = \frac{\sqrt{2}}{\rho P} d\overline{\zeta}. \tag{3.9}$$

It follows at once from this that, in terms of the NP operators Δ and δ,

$$\Delta\zeta = 0 = \delta\overline{\zeta},$$

while $\delta\zeta = -\frac{\overline{\rho}P}{\sqrt{2}}$.

We need covariant and contravariant expressions for the rest of the NP tetrad. We have

$$\begin{aligned} \Delta &= (\Delta u)\partial_u + (\Delta r)\partial_r, \\ \delta &= (\delta u)\partial_u + (\delta r)\partial_r - \frac{\overline{\rho}P}{\sqrt{2}}\partial_\zeta. \end{aligned}$$

From the commutator $[\Delta, D]$ (given in [16])

$$D\Delta u = 0$$

so that $\Delta u = X_6(u, \zeta, \overline{\zeta})$ for some function X_6, and analogously, from the commutator $[\delta, D]$ (using (3.4)) $\delta u = \overline{\rho} X_7$ in terms of another function X_7 of $(u, \zeta, \overline{\zeta})$. Since ℓ^a is null, we have

$$\ell_a dx^a = A du + B d\zeta + \overline{B} d\overline{\zeta}$$

for some real A and complex B (not to be confused with B appearing in (3.2)), and then normalization against m^a and n^a forces

$$AX_6 = 1\ , \quad A\overline{\rho}X_7 - B\frac{\overline{\rho}P}{\sqrt{2}} = 0\ ,$$

so that A and B are independent of r and can be found from $\tilde{\ell}$ at $\mathscr{I}^+$. There we have (3.3) so that, on $\mathscr{I}^+$,

$$\begin{aligned} \ell_a dx^a &= (\widetilde{L}_a + L\overline{\widetilde{M}}_a + \overline{L}\widetilde{M}_a)dx^a \\ &= du - \frac{\sqrt{2}L}{P}d\zeta - \frac{\sqrt{2}\overline{L}}{P}d\overline{\zeta}. \end{aligned} \tag{3.10}$$

Now we argue as for $m_a dx^a$: from what we have deduced already for $\ell_a dx^a$, we know that (3.10) holds in the interior. This gives the NP tetrad in the covariant form as

$$D = \partial_r \tag{3.11}$$

$$\Delta = \partial_u + H\partial_r \tag{3.12}$$

$$\delta = -\frac{\overline{\rho}P}{\sqrt{2}}(\partial_\zeta + \frac{\sqrt{2}L}{P}\partial_u - Q\partial_r) \tag{3.13}$$

where H and Q are still to be determined, and in the contravariant form as

$$\ell_a dx^a = du - \frac{\sqrt{2}L}{P}d\zeta - \frac{\sqrt{2}\overline{L}}{P}d\overline{\zeta} \tag{3.14}$$

$$n_a dx^a = dr + Qd\zeta + \overline{Q}d\overline{\zeta} - H\ell_a dx^a \tag{3.15}$$

$$m_a dx^a = \frac{\sqrt{2}}{\rho P}d\overline{\zeta}. \tag{3.16}$$

Once we have the radial dependence of Q and H, we have the radial dependence of the metric. From the $[\Delta, D]$ commutator we find

$$DH = -(\gamma + \overline{\gamma}),$$

while from (A.3f) and (A.4c)

$$D\gamma = \psi_2, \tag{3.17}$$

$$D\psi_2 = 3\rho\psi_2, \tag{3.18}$$

so that, by (3.4),

$$\psi_2 = \rho^3\psi_2^0, \tag{3.19}$$

$$\gamma = \gamma^0 + \frac{1}{2}\rho^2\psi_2^0, \tag{3.20}$$

where ψ_2^0 and γ^0 are independent of r. Therefore

$$H = H^0 - (\gamma^0 + \overline{\gamma}^0)r - \frac{1}{2}\rho\psi_2^0 - \frac{1}{2}\overline{\rho}\overline{\psi}_2^0, \tag{3.21}$$

where H^0 is independent of r; so (3.21) gives the radial dependence of H.

For Q, the commutator $[\delta, D]$ gives

$$-\frac{\overline{\rho}P}{\sqrt{2}}DQ = \overline{\alpha} + \beta,$$

while (A.3d) and (A.3e) can be integrated to give

$$\alpha = -\alpha^0\rho \ ; \quad \beta = -\beta^0\overline{\rho},$$

with α^0 and β^0 independent of r. Therefore

$$Q = Q^0 + \frac{\sqrt{2}}{P}(\overline{\alpha^0} + \beta^0)r, \tag{3.22}$$

with Q^0 independent of r.

For later use, we find the radial dependence of the remaining spin coefficients, μ and ν. For μ, we integrate (A.3h) to find

$$\mu = \mu^0\overline{\rho} + \frac{1}{2}\rho(\overline{\rho} + \rho)\psi_2^0, \tag{3.23}$$

where μ^0 independent of r. For ν, first from (A.4e), assuming $\Phi_{12} = 0$,

$$\psi_3 = \psi_3^0\rho^2 + \psi_3^1\rho^3 + \psi_3^2\rho^4,$$

where ψ_3^i are independent of r, and then from (A.3i),

$$\nu = \nu^0 + \psi_3^0\rho + \frac{1}{2}\psi_3^1\rho^2 + \frac{1}{3}\psi_3^2\rho^3, \tag{3.24}$$

where ν^0 is independent of r.

We now note the changes in the non-vacuum case. As in the Goldberg-Sachs theorem, we continue to insist on

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = 0 = \Lambda,$$

but allow the possibility of non-zero Φ_{11}, Φ_{12}, and Φ_{22} (in fact, Φ_{22} doesn't arise in the calculation). This changes some of the details above. We still have

$$\kappa = \sigma = \epsilon = \tau = \pi = \lambda = 0,$$

and (3.4), but (3.17)-(3.20) change. For the radial dependence of Φ_{11} we have equation (A.4i):

$$D\Phi_{11} = 2(\rho + \overline{\rho})\Phi_{11},$$

which integrates readily. With this, (A.4c) can be integrated for ψ_2, then (A.3f) for γ and then H obtained from the commutator $[\Delta, D]$. In place of (3.19)-(3.20) and (3.21), we find

$$\begin{aligned}
\Phi_{11} &= (\rho\overline{\rho})^2\Phi_{11}^0 &(3.25)\\
\psi_2 &= \rho^3\psi_2^0 + 2\rho^3\overline{\rho}\Phi_{11}^0 &(3.26)\\
\gamma &= \gamma^0 + \frac{1}{2}\rho^2\psi_2^0 + \rho^2\overline{\rho}\Phi_{11}^0 &(3.27)\\
H &= H^0 - (\gamma^0 + \overline{\gamma}^0)r - \frac{1}{2}\rho\psi_2^0 - \frac{1}{2}\overline{\rho}\overline{\psi}_2^0 - \rho\overline{\rho}\Phi_{11}^0. &(3.28)
\end{aligned}$$

where, as usual, quantities with a superscript zero are independent of r, and Φ^0_{11} is real. This is enough for the metric. For the spin-coefficient μ, (A.3h) now gives

$$\mu = \mu^0\overline{\rho} + \frac{1}{2}\rho(\overline{\rho}+\rho)\psi^0_2 + \rho^2\overline{\rho}\Phi^0_{11}. \tag{3.29}$$

The remaining spin-coefficient ν is altogether more complicated. We need to solve (A.4j) for Φ_{12}, then (A.4e) for ψ_3 and then (A.3i) for ν. The results are polynomials in ρ and $\overline{\rho}$, with coefficients constant along ℓ. We don't need the detailed expressions for these quantities, which can be found in [26]. For our purposes, the following suffices

$$\begin{aligned} \Phi_{21} &= O(|\rho|^3)\ , \\ \psi_3 &= O(|\rho|^2)\ , \\ \nu &= \nu^0 + O(|\rho|)\ . \end{aligned} \tag{3.30}$$

We are ready to prove now:

Proposition 3.3 *Let*

$$\begin{aligned} \mathscr{N} \;=\; \{p\in\mathscr{M}:\ & \textit{the null geodesic through } p \\ & \textit{tangent to } \ell(p) \textit{ has an end point on } S^+\}\ . \end{aligned}$$

There exist coordinates (u,r,ζ) parameterizing a neighborhood of $\mathscr{N}$, such that (u,ζ) coincide with Bondi coordinates on S^+, in which the metric takes the form $2\ell_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}$, with ℓ, n and m given by (3.9)-(3.16), with ρ given by (3.4), H given by (3.21), while Q is given by (3.22), where α^0, β^0, ψ^0_2, H^0, Q^0, L and Σ are smooth functions of u and ζ.

Proof: Transversality and smoothness of $\tilde{\ell}$ at S^+ imply that there exists a neighborhood of S^+ on which $\tilde{\ell}$ is transverse to $\mathscr{I}^+$, and the result follows from smoothness of ℓ together with the calculations above. □

Now we have the r-dependence of the metric. By construction, the coordinates u and ζ are good coordinates on $\mathscr{I}^+$ near S^+, while $r\Omega \to 1$. Rescaling the metric by r^{-2} and setting $R = r^{-1}$, we obtain for the asymptotic behavior

$$\widetilde{g} = R^2 g = 2(du - \sqrt{2}\frac{L}{P}d\zeta - \sqrt{2}\frac{\overline{L}}{P}d\overline{\zeta})(-dR + O(R)) - 4\frac{d\zeta d\overline{\zeta}}{P^2}(1+O(R^2)) \tag{3.31}$$

(recall that we shifted r to obtain $r^0 = 0$), which shows explicitly that the space-time is weakly asymptotically simple with this choice of rescaling.

Let S^- be obtained by flowing S^+ from $\mathscr{I}^+$ to $\mathscr{I}^-$ along $\tilde{\ell}$. We have:

Lemma 3.4 *S^- is a smooth acausal cross-section of $\mathscr{I}^-$, with both S^+ and S^- diffeomorphic to S^2.*

Proof: In the construction leading to (3.31) we consider instead $r\to-\infty$, taking $R = -r^{-1}$ to obtain a conformal completion $\overline{\mathscr{U}_i}$ at past infinity of the coordinate patch, say $\mathscr{U}_i$, constructed above. Consider the map ψ which to

$p \in S^+$ assigns the generator of $\mathscr{I}^- = \mathbb{R} \times S^2$ which is met by the maximally extended null geodesic tangent to $\tilde{\ell}$ and passing through p. Applying [5, Theorem 3.1] to $\overline{\mathscr{U}_i}$ we conclude that there exists a smooth local diffeomorphism from $\overline{\mathscr{U}_i}$ to $\widetilde{\mathscr{M}}$, so that S^- is a smooth immersed submanifold of $\mathscr{I}^-$; note, however, that S^- might fail to be embedded because some points of $\mathscr{I}^-$ could be met by more than one integral curve of $\tilde{\ell}$ emanating from S^+. In any case, we infer that ψ is a local diffeomorphism. By [12, Exercise 11-9, p. 253] ψ is a covering map, and since S^2 is simply connected it follows that ψ is a diffeomorphism, so $S^+ \approx S^2$, and S^- intersects every generator precisely once. As the only causal curves within $\mathscr{I}^-$ are the generators of $\mathscr{I}^-$, the result follows. □

As observed by Mason [14], one has

Proposition 3.5 *The Trautman-Bondi mass $m_{\mathrm{TB}}(S^+)$ of S^+ equals the negative of the Trautman-Bondi mass $m_{\mathrm{TB}}(S^-)$ of S^-.*

This will follow if the mass aspect has the same property. Mason suggests two proofs for this proposition: either via a direct check on the mass aspect or by exploiting an alternative formula for the Trautman-Bondi mass given in [20]. We shall present the first, exploiting a formula in [15] for the mass aspect. First we note that the correspondence $(u, r, \xi)_{NT} = (u, r, \zeta\sqrt{2})_{CT}$ relates our coordinates (subscript CT) to the ones used in [15] (subscript NT). Then the quantities $(L, H, W, m + iM, \Sigma, \hat{P})_{NT}$ arising in their metric are for us $(-\frac{L}{P}, -H, \frac{Q}{\sqrt{2}}, \psi_2^0, \Sigma, 1)_{CT}$, and finally their operator ∂ translates for us as

$$\partial = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \zeta} + \frac{L\sqrt{2}}{P} \frac{\partial}{\partial u} \right). \tag{3.32}$$

With these preliminaries, the Natorf-Tafel formula (47) of [15] for the integrand of the Trautman-Bondi mass (which we will refer to as the *mass aspect*; but note that this is *not* the original mass aspect function of [3, 21]) translates for us to

$$\mathcal{M} = \frac{1}{2}(\psi_2^0 + \overline{\psi}_2^0) + 3\Sigma\Sigma_{,u} + \frac{1}{2}(\tilde{\Delta} + 2)\eta - 2iP(\overline{L}_{,u}\partial\Sigma - L_{,u}\overline{\partial}\Sigma) , \tag{3.33}$$

where $\tilde{\Delta} = P^2(\partial\overline{\partial} + \overline{\partial}\partial)$ and

$$\eta = -\frac{1}{2}P^2 \left(\partial \left(\frac{\overline{L}}{P} \right) + \overline{\partial} \left(\frac{L}{P} \right) \right).$$

To investigate the mass aspect at $\mathscr{I}^-$, we define a time-reversed space-time for which the old $\mathscr{I}^-$ is now $\mathscr{I}^+$. With hatted quantities referring to the time-reversed space-time, we take coordinates $(\hat{u}, \hat{r}, \hat{\zeta}) = (-u, -r, \overline{\zeta})$. (The redefinition $r \to -r$ should be clear in our context; the need to replace ζ by $\overline{\zeta}$ arises then from elementary orientation considerations; the transition $u \to -u$ arises from the fact that we will be using on $\mathscr{I}^-$ a formula for the Trautman-Bondi mass which has been worked out at $\mathscr{I}^+$, and this requires a change of time-orientation). Then all our calculations so far can be repeated in the

tetrad $(\hat{\ell}, \hat{n}, \hat{m}) = (-\ell, -n, \overline{m})$. In particular $\hat{\rho}$ equals $-\overline{\rho}$, and tracing through the quantities in the mass aspect, we find $(\hat{\Sigma}, \hat{L}, \hat{\eta}, \hat{\eth}, \hat{\psi}_2^0) = (\Sigma, -\overline{L}, -\eta, \overline{\eth}, -\overline{\psi}_2^0)$. Using these in (3.33), we see that, as desired, $\mathcal{M}$ changes sign. Now all quantities appearing in (3.33) are constant along ℓ, and we conclude that the mass aspect at S^- is the negative of the mass aspect at S^+. □

4 Spacelike hypersurfaces, a rigid positive energy theorem

Choose a cut S^+ of $\mathscr{I}^+$ and consider the associated null boundary

$$\mathscr{N} := \dot{J}^-(S^+, \mathscr{M}) \,, \tag{4.1}$$

then $\mathscr{N}$ is an achronal hypersurface generated by null geodesics orthogonal to S^+. Further there exists a neighborhood $\mathscr{O}$ of $\mathscr{I}^+$ such that $\mathscr{N} \cap \mathscr{O}$ is smooth. If we assume $(\mathscr{M}, {}^4g)$ to be globally hyperbolic, there exists a time-function τ on $\mathscr{M}$ with the property that its level sets,

$$\mathring{\mathscr{S}}_{\tau_0} := \{\tau = \tau_0\} \,,$$

are smooth spacelike Cauchy surfaces (compare [2]). Define

$$\mathscr{N}_\tau := \dot{J}^-(S^+) \cap \mathring{\mathscr{S}}_\tau \,;$$

note that the intersection is transverse. Since $\mathscr{M} = \cup_\tau \mathring{\mathscr{S}}_\tau$, we have that $\cup_\tau \mathscr{N}_\tau = \mathscr{N}$. This, together with Dini's theorem, shows that there exists τ_0 such that $\mathscr{N}_{\tau_0} \subset \mathscr{O}$, thus $\mathscr{N}_{\tau_0}$ is a smooth sphere.

For $\epsilon > 0$ let ${}^4\tilde{g}_\epsilon$ be a family of smooth Lorentzian metrics on $\widetilde{\mathscr{M}}$ such that ${}^4\tilde{g}_\epsilon$ converges to ${}^4\tilde{g}$ on compact subsets of $\widetilde{\mathscr{M}}$ as ϵ goes to zero in the C^∞ topology, with the property that all vectors which are null for ${}^4\tilde{g}_\epsilon$ are spacelike for ${}^4\tilde{g}$. By continuous dependence of geodesics upon the metric, for $\epsilon > 0$ small enough all null ${}^4\tilde{g}_\epsilon$-geodesics normal to $\mathscr{I}^+$ intersect $\mathring{\mathscr{S}}_{\tau_0}$ in a smooth sphere $\mathring{N}$, with the corresponding hypersurface $\mathscr{N}^\epsilon$, defined as in (4.1) using the metric ${}^4\tilde{g}_\epsilon$, being smooth in its portion which is bounded by S^+ and by $\mathring{N}$; call this region $\mathscr{S}_{\mathrm{ext}}$. The Cauchy surface $\mathring{\mathscr{S}}_{\tau_0}$ is contractible by one of the hypotheses of Theorem 1.1, or by [17, Section 5] if full asymptotic simplicity is assumed. Simple connectedness of $\mathring{\mathscr{S}}_{\tau_0}$ and elementary intersection theory show that $\mathring{N}$ separates $\mathring{\mathscr{S}}_{\tau_0}$ into two components. From the Hurewicz isomorphism theorem [22, Chapter 7, Section 5] we further conclude that $H_2(\mathring{\mathscr{S}}_{\tau_0})$ is trivial, which implies that one of the components separated by the sphere $\mathring{N}$, say $\mathscr{K}$, is compact. Set

$$\mathscr{S} = \mathscr{K} \cup \mathscr{S}_{\mathrm{ext}} \,,$$

then $\mathscr{S}$ is a piecewise differentiable ${}^4g$–spacelike hypersurface which is the union of a compact set and of an asymptotic region extending to $\mathscr{I}^+$. Smoothing out the corner at $\mathring{N}$ one obtains a smooth hypersurface, still denoted by $\mathscr{S}$. Next, the formulae of [6, Appendix C.3] show how to make a small deformation of $\mathscr{S}$

near $\mathscr{I}^+$ to obtain a hypersurface on which the induced metric asymptotes to a hyperbolic one, as needed for the proof of positivity of mass of [7]. Finally, we let $\widehat{\mathscr{S}}$ be the universal cover of $\mathscr{S}$, then $\widehat{\mathscr{S}}$ is complete, with one or more asymptotically hyperbolic ends.[10]

By the positive energy theorem of [7] applied to a chosen asymptotic end of $\widehat{\mathscr{S}}$ we conclude that the Trautman-Bondi mass associated with this end is non-negative. An identical construction starting from S^- shows that $m_{\mathrm{TB}}(S^-) \geq 0$. From Proposition 3.5 we infer that

$$m_{\mathrm{TB}}(S^+) = 0 \; .$$

We continue with an investigation of the consequences of the Witten-type proof of the positive energy theorem on $\widehat{\mathscr{S}}$. Let $\mathring{\psi}$ be a Dirac spinor which is parallel with respect to the spin-connection associated with the Minkowski metric, such that the resulting Killing vector in Minkowski space-time $\mathbb{R}^{1,3}$ equals ∂_t. Now, when $m = 0$, the proof of the positive energy theorem in [7] shows that the space-time metric ${}^4g$ is flat along $\widehat{\mathscr{S}}$, and that there exists a spinor ψ, solution of the Witten equation, such that $\psi = \mathring{\psi} + \chi$, with χ in a weighted Sobolev space obtained by completing C_0^∞ with respect to the norm

$$\sqrt{\int |D\chi|^2 d\mu_g} \; .$$

Furthermore, ψ is parallel with respect to the space-time connection ∇ associated to the initial data set $(\widehat{\mathscr{S}}, g, K)$,

$$\nabla_i \psi := D_i \psi + \frac{1}{2} K_{ij} \gamma^j \gamma^0 \psi = 0 \; . \tag{4.2}$$

Let (V, Y) be the KID defined by ψ,

$$V := \langle \psi, \psi \rangle \; , \quad Y := \langle \psi, \gamma^0 \gamma^j \psi \rangle e_j \; .$$

Equation (4.2) implies that (V, Y) is parallel, in the following sense:

$$D_i V = K_{ij} Y^j \; , \quad D_i Y_j = V K_{ij} \; . \tag{4.3}$$

It follows that the Lorentzian norm squared $V^2 - |Y|_g^2$ of (V, Y) is constant on $\mathscr{S}$,

$$D_i \Big(V^2 - |Y|_g^2 \Big) = 0 \; . \tag{4.4}$$

It should follow from the methods in [1] that this norm is strictly positive by choice of $\mathring{\psi}$, so that the associated Killing vector is timelike; however, an argument which avoids the heavy machinery of the last reference proceeds as follows: If $V^2 - |Y|_g^2 = 0$, we choose a different asymptotic value of $\mathring{\psi}$. If the new resulting Killing vector is timelike we are done, otherwise there is a linear

[10] Conceivably one can infer at this stage, from the results in [17], that $\mathscr{S}$ is simply connected; however, the argument that follows sidesteps this issue.

combination of the new Killing vector and of the old one which is timelike, and satisfies (4.3), leading to a timelike Killing vector for which (4.4) holds as well.

We consider the Killing development $(\mathring{\mathscr{M}},\mathring{g})$ defined by $(\widehat{\mathscr{S}},V,Y)$, thus $\mathring{\mathscr{M}}$ is $\mathbb{R}_t\times\widehat{\mathscr{S}}$ with metric

$$\mathring{g}=V^2dt^2-g_{ij}(dx^i+Y^idt)(dx^j+Y^jdt)\ ,$$

with Killing vector $X=\partial_t$. Letting

$$\exp(\mu):=\ {}^4g(X,X)=V^2-|Y|^2_g\ , \tag{4.5}$$

we rewrite the Killing development metric $\mathring{g}$ in the following form,

$$\mathring{g}=\exp(\mu)(dt+\theta_idx^i)^2-h\ , \tag{4.6}$$

as needed in Lemma 3.11 of [8], where the Riemannian metric h is related to the initial data metric g by the equation

$$h_{ij}=g_{ij}+\exp(\mu)\theta_i\theta_j\ , \tag{4.7}$$

and

$$\theta_i=-e^{-\mu}g_{ij}Y^j\ . \tag{4.8}$$

To apply that last Lemma, we need to verify that the metric h in (4.6) is complete, and that the h-length of θ is uniformly bounded on $\widehat{\mathscr{S}}$. Now, the hyperbolic asymptotics of g, together with compactness of $\widehat{\mathscr{S}}\cup\dot{\widehat{\mathscr{S}}}$ and the Hopf-Rinow theorem, imply completeness of $(\widehat{\mathscr{S}},g)$. Since the last term in (4.7) gives a non-negative contribution on any given vector, completeness of $(\widehat{\mathscr{S}},h)$ follows from that of $(\widehat{\mathscr{S}},g)$.

Next, it follows from (4.4) that $\exp(\mu)$ is constant over $\widehat{\mathscr{S}}$. Further,

$$h^{ij}=g^{ij}-\frac{\exp(-\mu)}{1+\exp(-\mu)|Y|^2_g}Y^iY^j\ ,$$

so that

$$|\theta|^2_h=h^{ij}\theta_i\theta_j=\frac{\exp(-2\mu)|Y|^2_g}{1+\exp(-\mu)|Y|^2_g}\le\exp(-\mu)=:C\ ,$$

which establishes the desired uniform bound on $|\theta|_h$.

From [8, Lemma 3.11] we conclude that the Killing development $(\mathring{\mathscr{M}},\mathring{g})$ of $(\widehat{\mathscr{S}},V,Y)$ is geodesically complete. Since $\widehat{\mathscr{S}}$ is simply connected, so is $\mathring{\mathscr{M}}\approx\mathbb{R}\times\widehat{\mathscr{S}}$. The Lorentzian version of the Hadamard-Cartan theorem [19, Proposition 23, p. 227] implies that $(\mathring{\mathscr{M}},\mathring{g})$ is $\mathbb{R}^{1,3}$. In particular $\widehat{\mathscr{S}}$ is a hyperboloidal hypersurface in $\mathbb{R}^{1,3}$, and hence has only one asymptotically hyperbolic end. But if $\mathscr{S}$ were not simply connected, $\widehat{\mathscr{S}}$ would have had more than one such end. We conclude that $\widehat{\mathscr{S}}=\mathscr{S}$.

By hypothesis $(\mathscr{M},{}^4g)$ satisfies the dominant energy condition, hence the domain of dependence $\mathscr{D}(\mathscr{S},\mathscr{M})$ in the original space-time $\mathscr{M}$ is vacuum by [11, Section 4.3]. From [4] we conclude that $\mathscr{D}(\mathscr{S},\mathscr{M})$ is isometrically diffeomorphic to a globally hyperbolic subset of the domain of dependence $\mathscr{D}(\mathscr{S},\mathring{\mathscr{M}})$ in the

Killing development. This is a bijection when $(\mathscr{M}, {}^4g)$ is both past and future asymptotically simple, otherwise $\mathscr{D}(\mathscr{S},\mathscr{M})$ couldn't be future null geodesically complete.

For the record, the above establishes the following rigidity statement (compare Theorems 5.4 and 5.7 of [7]; the reader is referred to this last reference for precise definitions):

THEOREM 4.1 *Let μ, respectively J^i, denote the energy density, respectively the momentum density, of an initial data set $(\mathscr{S}, g, K)$. Suppose that $(\mathscr{S}, g)$ is geodesically complete without boundary, and that $\mathscr{S}$ contains an end which is $C^4 \times C^3$, or C^1 and polyhomogeneously, compactifiable and asymptotically CMC, with energy-momentum density decaying fast enough. If*

$$\sqrt{g_{ij}J^iJ^j} \le \mu \ , \tag{4.9}$$

and if the Trautman-Bondi mass of $\mathscr{S}$ vanishes, then $(\mathscr{S}, g, K)$ can be realized by embedding $\mathscr{S}$ into Minkowski space-time.

If the initial data set is known to be vacuum near the conformal boundary from the outset, or to satisfy a set of equations which are well behaved under singular conformal transformations such as, e.g., the Einstein–Maxwell or Einstein–Yang-Mills equations, then the restriction that the data be asymptotically CMC is not needed. □

REMARK 4.2 It is still an open question whether a null Trautman-Bondi energy-momentum is compatible with the remaining hypotheses above; it would be of interest to settle that.

We continue by pushing S^+ slightly down the generators of $\mathscr{I}^+$, to conclude that the space-time metric is flat in a neighborhood of S^+. By [5] we conclude that

PROPOSITION 4.3 *There exists a neighborhood of S^+ which is isometrically diffeomorphic to a neighbourhood of a spherical cut of $\mathscr{I}^+$ in Minkowski space-time. Moreover the space-time metric is flat to the future of any spacelike hypersurface spanned by S^+.* □

Recalling that the congruence generated by ℓ has nowhere vanishing twist (see Proposition 3.1), Theorem 1.1 follows now from Proposition 4.4: □

PROPOSITION 4.4 *There exists no smooth, null, geodesic congruence defined in a neighborhood of a cross-section S^+ of the Minkowskian $\mathscr{I}$ which is shear-free, transverse to S^+, and has nowhere vanishing twist.*

PROOF: Any smooth, null-geodesic congruence near a Minkowskian $\mathscr{I}^+$ defines a spin-weight one function L as in (3.3) which is smooth on $\mathscr{I}^+$ in a neighbourhood of the cut $S^+ = \{u = 0\}$. By [13, Equation (2.24)] the shear-free condition in Minkowski space is equivalent to

$$\eth L + L\dot{L} = 0 \ , \tag{4.10}$$

where the dot stands for $\partial/\partial u$ and $\eth$ is the eth-operator of Newman and Penrose. Then Σ in (3.4) is given by (see [13, Equation (2.17)])

$$\Sigma = \frac{i}{2}(\bar{\eth} L + \overline{L}\dot{L} - \eth\overline{L} - L\dot{\overline{L}}) \,. \tag{4.11}$$

Now consider $F := L\overline{L}$ restricted to the cut $u = 0$. Clearly F has a maximum on this sphere, and at the maximum its gradient vanishes, so at any maximum

$$0 = \eth F = (\eth L)\overline{L} + L(\eth\overline{L}) = L(\eth\overline{L} - \overline{L}\dot{L}), \tag{4.12}$$

using (4.10) to eliminate $\eth L$. If $L \equiv 0$ on the cut $\{u = 0\}$ then $\Sigma = 0$ throughout the cut by (4.11), and we are done. Otherwise, at a maximum of F, L does not vanish so that the second factor in (4.12) must. But by (4.11) this forces Σ to vanish there. □

Proof of Theorem 2.1: Under (2.1), the argument of the proof of Theorem 1.1 with S^+ replaced by S_i^+ carries through with minor modifications. Indeed, the cross-sections S_i^- are smooth acausal cross-sections of $\mathscr{I}^-$ as before, because they are constructed by flowing along the geodesics which start at $\mathscr{I}^+$, and those do not care about smoothness of ℓ as a vector field on $\mathscr{M}$. Next, the argument that the Bondi mass aspect changes sign remains valid for those members of the congruence which have end points on $\mathring{S}_i^+$. But the Bondi mass aspect is a smooth function both on S_i^+ and S_i^-, and the density hypothesis (2.1) guarantees that the corresponding subset of S_i^- is dense. Continuity allows us to conclude, as before, that the Trautman-Bondi mass changes sign when replacing S_i^+ with S_i^-.

Let $\mathscr{S}_i$ be a hypersurface $\mathscr{S}$ as in the proof of Theorem 1.1 with S^+ there replaced by S_i^+. We have shown so far that $\mathscr{S}_i$ is a hyperboloidal hypersurface in Minkowski space-time and, since it has no edge, its future (whether in Minkowski space-time or in $\mathscr{M}$) coincides with its future domain of dependence there:

$$\mathscr{D}^+(\mathscr{S}_i, \mathscr{M}) = J^+(\mathscr{S}_i, \mathscr{M}) \,. \tag{4.13}$$

Now, by asymptotic simplicity, every generator of the Cauchy horizon[11] $\dot{\mathscr{D}}^-(\mathscr{S}_i)$ has a future end point on S_i^+. This implies that

$$\dot{\mathscr{D}}^-(\mathscr{S}_i, \mathscr{M}) = \dot{J}^-(S_i^+, \widetilde{\mathscr{M}}) \cap \mathscr{M} \,. \tag{4.14}$$

We continue by showing that

$$\underbrace{\cup_{i\in\mathbb{N}}\mathscr{D}(\mathscr{S}_i, \mathscr{M})}_{=:\mathscr{U}} = \mathscr{M} \,. \tag{4.15}$$

Suppose that this is not the case, then there exists a sequence of points $p_i \in \dot{\mathscr{D}}^-(\mathscr{S}_i)$ which converges to a point p belonging to the boundary $\dot{\mathscr{U}}$ of $\mathscr{U}$. Let $\dot{\gamma}_i$ be the vector tangent to a generator of $\dot{\mathscr{D}}^-(\mathscr{S}_i)$ at p_i, normalized to unit norm with respect to an auxiliary Riemannian metric. Passing to a subsequence if

[11] There are two conventions for defining $\mathscr{D}(\mathscr{S})$, we use the one in which inextendible timelike curves are required to intersect $\mathscr{S}$ precisely once.

necessary, the sequence $(\dot\gamma_i)$ converges to a null vector $\dot\gamma$ at p. Let γ be a null geodesic through p with tangent $\dot\gamma$ there, maximally extended in $\widetilde{\mathscr{M}}$, then γ meets $\mathscr{I}^+$ at some point q.

Without loss of generality, passing to a subsequence if necessary, we can assume that

$$S^+_{i-1} \subset J^+(S^+_i, \widetilde{\mathscr{M}}) \ .$$

Since $\mathscr{I}^+ \subset \cup_i J^+(S^+_i)$ there exists i_0 such that $q \in J^+(S^+_{i_0})$. Then γ intersects $\mathscr{D}(\mathscr{S}_i)$ for every $i < i_0$, and since $p \notin \mathscr{D}(\mathscr{S}_i)$ the null geodesic γ, when followed to the past starting from q, has to intersect $\dot J^-(S^+_i)$ before reaching p, compare (4.14). But the γ_i's accumulate at γ as i tends to infinity, so that there exists $i_1 > i_0+1$ so that (by continuous dependence of solutions of ODE's upon initial values) the geodesic $\gamma_{i_1} \subset \dot J^-(S^+_{i_1})$ intersects $\dot J^-(S^+_{i_0+1})$. This is, however, not possible since $\dot J^-(S^+_{i_0+1})$ is strictly interior to $J^+(\dot J^-(S^+_{i_1}))$. We conclude that (4.15) holds, and therefore ${}^4g$ is flat.

Summarising, $(\mathscr{M}, {}^4g)$ is a simply connected, flat, null geodesically complete manifold. Theorem 2.1 follows now from Proposition 4.5 below.[12] $\square$

PROPOSITION 4.5 *Let* $n \geq 1$. *The only* $(n+1)$*-dimensional simply connected, flat, null* <u>*or*</u> *timelike geodesically complete Lorentzian manifold* $(\mathscr{M}, g)$ *is, up to isometric diffeomorphism, the Minkowski space-time* $\mathbb{R}^{1,n}$.

PROOF: Since g is flat, the dimension of the set of germs of locally defined Killing vector fields is the same at every point. A theorem of Nomizu [18] shows then that every local Killing vector extends to a globally defined one. By [10, Lemma 1], all Killing vector fields[13] are complete. But, in a flat space-time, affinely parameterized geodesics are orbits of translational Killing vectors, hence $(\mathscr{M}, {}^4g)$ is geodesically complete. The result follows now from the Hadamard-Cartan theorem.

$\square$

5 Concluding remarks

One would like to remove all restrictive hypotheses of Theorem 2.1 and assert that the only algebraically special vacuum asymptotically simple space-time is the Minkowski one. Any proof of this, in a setting where the set

$$\mathscr{V} \ := \ \{p \in \mathscr{I}^+ | \ p \text{ is an end-point of an integral curve } \gamma \text{ of } \ell\} \subset \mathscr{I}^+$$

does *not* cover a dense subset of some sequence of cross-sections of $\mathscr{I}^+$, has to use arguments going beyond those indicated by Mason. On the other hand,

[12]We are grateful to a referee for a suggestion leading to Proposition 4.5.

[13]The hypothesis that the Killing vector is timelike, made elsewhere in [10], is not used in the proof, which goes through unchanged with one exception: when $n = 1$, and the manifold is assumed to be null geodesically complete, and the Killing orbit is null. But in this case the orbit is a null geodesic, so null geodesic completeness implies completeness of that orbit trivially.

one could expect that some version of the current argument should apply if the last density property holds. However, attempts to include such situations face several difficulties. Suppose, for instance, that S is a cross-section of $\mathscr{I}^+$ such that $\mathscr{V} \cap S$ is dense in S. Now, Mason's construction requires flowing from $\mathscr{V} \cap S$ to the past along the integral curves of $\tilde{\ell}$. Since $\mathscr{V} \cap S$ is not compact anymore, the geometry of the resulting subset S^- of $\mathscr{I}^-$ is not clear: By causality considerations, S^- will be bounded to the future on $\mathscr{I}^-$, however, it could very well be unbounded to the past. Regardless of that issue, the closure $\overline{S^-}$ of S^- might fail to be differentiable. Finally, S^- might develop self-intersections. In all those cases a useful notion of mass of S^- is not clear, and certainly no suitable positivity theorem is available. Similar problems concerning the geometry of S_i^- could occur in those space-times in which $\mathscr{I}^-$ is not diffeomorphic to $\mathbb{R} \times S^2$; while we are not aware of any such asymptotically simple examples, their existence has not been ruled out so far (strong causality at $\mathscr{I}^-$ must then necessarily fail, compare [17]).

In this context the following example is rather instructive: Consider a cut S of $\mathscr{I}^+$ in Minkowski or Schwarzschild space-time given by the equation $u = \alpha$, then the integrand of the Trautman-Bondi mass of S equals

$$\frac{m}{4\pi} + \frac{1}{16\pi}\Delta_2(\Delta_2 + 2)\alpha \ ,$$

where m is the Schwarzschild mass parameter (which we set to zero in the Minkowski case), while Δ_2 is the Laplacian on S_2 (see, e.g., [6, p. 136]). Now, with a little work one finds that for any $c \in \mathbb{R}$ the function

$$\alpha_c = \frac{c}{4}\Big(\cos\theta \ln\tan(\theta/2) - 2\ln\sin\theta \Big)$$

is a solution of

$$\Delta_2(\Delta_2 + 2)\alpha_c = c \tag{5.1}$$

away from the north and south poles. We can add to α_c elements of the kernel of the operator appearing in (5.1) which, when allowing functions which are singular at the poles, contains $\ln\tan(\theta/2)$. By adding to α_c an appropriate multiple of this last function one can obtain a function $\alpha_{c,S}$ which solves (5.1) away from the south pole, as well as a function $\alpha_{c,N}$ which is a solution away from the north one

$$\alpha_{c,N} = \alpha_c - \frac{c}{4}\ln\tan(\theta/2) \ , \qquad \alpha_{c,S} = \alpha_c + \frac{c}{4}\ln\tan(\theta/2) \ .$$

The graph $\{u = \alpha\}$ of each of these functions provides thus an example of an embedded smooth submanifold of $\mathscr{I}^+$ (which fails to be a cut of $\mathscr{I}^+$ because it misses one generator) with Trautman-Bondi mass m_{TB}, when naively defined as the integral of the mass aspect function, being an affine function of c, in particular both the mass aspect and m_{TB} can be negative.

A piecewise smooth, non-differentiable, but (uniformly) Lipschitz continuous cross-section of the Minkowskian $\mathscr{I}^+$, with mass aspect function which is everywhere negative except at the equator where it is not defined, can be constructed by using the function

$$\alpha = \max(\alpha_{-1,S}, \alpha_{-1,N}) \ .$$

The reader may readily devise a similar example in Schwarzschild space-time, or in any space-time with a complete $\mathscr{I}^+$ in which the relevant functions are uniformly bounded over $\mathscr{I}^+$.

ACKNOWLEDGEMENTS We acknowledge useful discussions with, or comments from, M. Anderson, G. Galloway, W. Natorf, E.T. Newman, J. Tafel, and A. Trautman.

A Smoothness of ℓ for non-branching metrics

Let ℓ be the field of principal null directions of the Weyl tensor, normalized so that $\nabla_\ell \ell = 0$. In this appendix we wish to prove that ℓ is smooth on the set where the Weyl tensor is non-branching, as defined in the introduction; thus either of type II or D throughout the set, or type III throughout the set, or type N throughout. As already mentioned, in the type II or D case we allow the type to change from point to point, as long as the Weyl tensor remains in the II or D class.

Since the claim is local, it is sufficient to establish the result in a neighborhood of a point. So let o^A, ι^A be any local basis of the space of two-component spinors near p, and let ψ_{ABCD} be the Weyl spinor. Then, *by definition*, the Weyl tensor is type II or D if at least one of the solutions of the equation

$$\begin{aligned} 0 &= P(\lambda) := \psi_{ABCD}(\lambda\iota^A + o^A)(\lambda\iota^B + o^B)(\lambda\iota^C + o^C)(\lambda\iota^D + o^D) \\ &\equiv \psi_4\lambda^4 + \psi_3\lambda^3 + \psi_2\lambda^2 + \psi_1\lambda + \psi_0 \end{aligned} \tag{A.1}$$

corresponds to a zero which is exactly of second order. The associated principal null direction is (whatever the type) determined by the null vector $(\lambda\iota_A + o_A)(\overline{\lambda}\overline{\iota}_{A'} + \overline{o}_{A'})$. So smoothness of ℓ near p, for a smooth metric, will be proved if we show that the solution λ of (A.1) depends smoothly upon the coefficients ψ_i appearing in (A.1). We will actually show that λ is an analytic function of the coefficients, see Proposition A.1 below, so ℓ will be analytic if the Weyl tensor is. The analysis applies regardless of the order of the remaining roots of (A.1), which explains why the argument covers both the II and D Petrov-types (recall that type II is defined by requiring the remaining zeros to be simple, while type D correspond to a second order zero for the other root).

Similarly we define the Weyl tensor to be of type III throughout a set $\mathscr{U}$ if one of the solutions of (A.1) corresponds to a zero of exactly third order throughout $\mathscr{U}$; smoothness of the associated vector field ℓ follows then from Proposition A.2 below with $k = 3$. Finally type N is defined by requiring P to have one single zero of order four, and smoothness is a consequence of Proposition A.2 with $k = 4$.

We start by noting that, by passing to a different basis of the space of spinors if necessary, we can assume ψ_4 is non-zero at p. Indeed, suppose that ψ_4 is zero in any basis at p, then also $\psi_0 = 0$ for any basis at p, which implies $P(\lambda) = 0$ for all λ. It follows that $\psi_i = 0$ for all $i \in \{0, \dots, 4\}$, hence $\psi_{ABCD} = 0$ at p, thus the Weyl tensor is of type 0 there, contradicting our hypothesis that the Weyl tensor is non-branching on the set under consideration. From now on

we choose any basis so that $\psi_4(p) \neq 0$, but then by continuity there exists a neighborhood $\mathscr{V}_p$ of p on which ψ_4 has no zeros. All remaining considerations are restricted to $\mathscr{V}_p$, which involves no loss of generality since p is arbitrary within the non-branching set.

Dividing by ψ_4, we are led to study the equation

$$0 = \lambda^N + \sum_{i=0}^{N-1} \alpha_i \lambda^i \equiv W(\lambda) \ , \tag{A.2}$$

with smooth complex coefficients α_i (in the case of current interest, $\alpha_i = \psi_i/\psi_4$, and $N = 4$). Then λ is a zero of order two if and only if

$$W(\lambda) = W'(\lambda) = 0 \ , \quad \text{ but } \ W''(\lambda) \neq 0 \ .$$

We need to analyse the dependence of λ upon the coefficients α_i of (A.8). Consider, first, the equation

$$W'(\lambda) = 0 \ ; \tag{A.3}$$

the holomorphic implicit function theorem shows that (A.3) defines an analytic function $\lambda \equiv \lambda(\alpha_i)$ on the set

$$\mathscr{U}_2 := \{W''(\lambda) \neq 0 \ , \ \lambda \in \mathbb{C} \ , \ (\alpha_i) \in \mathbb{C}^N\} \subset \mathbb{C}^{N+1} \ , \tag{A.4}$$

with

$$\frac{\partial \lambda}{\partial \alpha_i} = -\frac{i\lambda^{i-1}}{W''(\lambda)} \ . \tag{A.5}$$

Next, let the function $\varphi : \mathscr{U}_2 \to \mathbb{C}$ be defined as $\varphi = W(\lambda(\alpha_i))$, by definition we have $W'(\lambda(\alpha_i)) = 0$ so that

$$d\varphi = \frac{\partial \varphi}{\partial \alpha_i} d\alpha_i = \Big(\underbrace{W'(\lambda)}_{=0} \frac{\partial \lambda}{\partial \alpha_i} + \lambda^i\Big) d\alpha_i = d\alpha_0 + \lambda d\alpha_1 + \ldots + \lambda^{N-1} d\alpha_{N-1} \ . \tag{A.6}$$

It follows that $d\varphi$ has no zeros on $\mathscr{U}_2$, hence $\{W(\lambda) = 0\}$ is an analytic submanifold of $\mathscr{U}_2$.

We have thus shown

Proposition A.1 *The set*

$$\mathscr{V}_2 := \{\alpha_i : \ W(\lambda) = W'(\lambda) = 0 \ , \ W''(\lambda) \neq 0 \ \ \textit{for some} \ \ \lambda \in \mathbb{C}\} \subset \mathbb{C}^N$$

is an analytic submanifold of co-dimension one in $\mathbb{C}^N$, with λ being an analytic function on $\mathscr{V}_2$.

The above generalizes immediately to zeros of W which are *exactly of order* k: indeed, set

$$\begin{aligned} \mathscr{V}_k \ := \ & \Big\{\alpha_i : \ \exists \ \lambda \in \mathbb{C} \text{ such that } W^{(i)}(\lambda) = 0 \ , \ i = 0, \ldots, k-1 \ , \\ & \text{but } \ W^{(k)}(\lambda) \neq 0\Big\} \subset \mathbb{C}^N \ . \end{aligned} \tag{A.7}$$

Then the equation $W^{(k-1)}(\lambda) = 0$ defines a smooth function λ on $\mathscr{V}_k$ by the implicit function theorem, using an obvious generalization of (A.5), and for $k = 1$ we are done. Otherwise consider the map $\phi = (\phi^i) : \mathscr{V}_k \to \mathbb{R}^k$, where

$$\phi^i = W^{(i)}(\lambda) \ , \quad i = 0, \ldots, k-1 \ .$$

On the preimage $\phi^{-1}(\{0\})$ we have, as in (A.6), $\partial\phi^j/\partial\alpha_i = i(i-1)\cdots(i-j)\lambda^{i-j}$, so that the last k columns of the Jacobi matrix take the form

$$\begin{pmatrix} \lambda^{k-1} & \lambda^{k-2} & \cdots & 0! \\ (k-1)\lambda^{k-2} & \cdots & 1! & 0 \\ \vdots & \iddots & 0 & 0 \\ (k-1)! & 0 & 0 & 0 \end{pmatrix} ,$$

the determinant of which is clearly non-vanishing. By the rank theorem one concludes that:

Proposition A.2 *The set $\mathscr{V}_k$ is an analytic submanifold of co-dimension $k-1$ in $\mathbb{C}^N$, with λ being an analytic function on $\mathscr{V}_k$.*

Remark A.3 Identical arguments apply to polynomials with real coefficients, $\mathbb{C}$ being replaced by $\mathbb{R}$ and "analytic" being replaced by "real analytic" both in the statements and in the proofs.

The argument just given also settles the following closely related question: consider a smooth function $A : \mathscr{U} \to \mathrm{End}(\mathbb{C}^N)$ or $A : \mathscr{U} \to \mathrm{End}(\mathbb{R}^N)$, defined on an open subset $\mathscr{U}$ of $\mathbb{R}^n$, with the property that for all $p \in \mathscr{U}$ the dimension of the associated eigenspace equals k. We further assume that A is hermitian in the complex case, or symmetric in the real one. We claim that the function which to $p \in \mathscr{U}$ assigns the associated k-dimensional eigenspace is a smooth function on $\mathscr{U}$.[14] In order to see this, let λ be a solution of the characteristic equation,

$$0 = \lambda^N + \sum_{i=1}^{N-1} \alpha_i \lambda^i \equiv W(\lambda) := \det(A - \lambda \mathrm{Id}) \ . \tag{A.8}$$

Then λ will have algebraic multiplicity k if and only if the α_i's belong to the set $\mathscr{V}_k$ of Proposition A.1. Composing with the map which to A assigns its symmetric polynomials α_i, and using Proposition A.2, we conclude that λ is a smooth function on $\mathscr{U}$ (analytic if A is). This allows us to show that:

Proposition A.4 *The k-dimensional eigenspaces are smooth functions on $\mathscr{U}$, analytic if A is.*

Proof: Let $p_0 \in \mathscr{U}$ and let $A_0 = A(p_0)$, $\lambda_0 = \lambda(p_0)$, thus there exist k linearly independent vectors $e_i \in \mathbb{C}^N$ such that

$$(A_0 - \lambda_0 \mathrm{Id})e_1 = \cdots = (A_0 - \lambda_0 \mathrm{Id})e_k = 0 \ .$$

[14] The question of multiple principal directions of the Weyl tensor, discussed at the beginning of this section, can also be formulated as such a problem [23].

We can complete $\{e_i\}_{i=1}^k$ to a basis $\{e_i\}_{i=1}^N$ of $\mathbb{C}^N$. In this basis any $A = A(p)$ can be written as

$$A = \lambda \mathrm{Id} + \begin{pmatrix} B & C \\ C^\dagger & E \end{pmatrix} , \quad \text{while} \;\; A_0 = \lambda_0 \mathrm{Id} + \begin{pmatrix} 0 & 0 \\ 0 & E_0 \end{pmatrix} ,$$

where B is a $k \times k$ matrix, with $\lambda = \lambda(p)$, and with B, C, and E being analytic functions of A, hence smooth in p (analytic if A is). Since $\dim \mathrm{Ker}(A_0 - \lambda_0 \mathrm{Id}) = k$ we have $\det E_0 \neq 0$, hence there exists a neighborhood of A_0 on which $\det E \neq 0$. For p within this neighborhood set $X_i = e_i + \hat{X}_i$, where the vectors $\hat{X}_i \in \mathrm{Vect}\{e_{k+1}, \ldots, e_N\}$ are given by

$$\hat{X}_1 = -E^{-1} C^\dagger e_1 \; , \; \ldots \; , \; \hat{X}_k = -E^{-1} C^\dagger e_k \; .$$

Clearly the X_i's are analytic functions of A, thus smooth (analytic if A is) in p. As $\mathrm{Ker}(A - \lambda \mathrm{Id})$ has dimension precisely k throughout $\mathscr{U}$ by hypothesis, it easily follows that the X_i's span $\mathrm{Ker}(A - \lambda \mathrm{Id})$. □

B Rescalings, ρ and smooth extendibility of $\tilde{\ell}$

Throughout this appendix the symbol ℓ denotes a vector field satisfying $\nabla_\ell \ell = 0$ together with (1.1). We assume that the Ricci tensor of $(\mathscr{M}, g)$ satisfies the conditions spelled out in the last part of Theorem 1.1. The aim here is to prove the following:

Theorem B.1 *Suppose that ℓ is smooth on the intersection $\mathscr{U} \cap \mathscr{M}$ of a neighborhood $\mathscr{U}$ of $\mathscr{I}^+$ with $\mathscr{M}$, and let*

$$\begin{aligned} \mathscr{V} &:= \{p \in \mathscr{I}^+ |\; p \textit{ is an end-point of an integral curve } \gamma \textit{ of } \ell\} \subset \mathscr{I}^+ , \\ \mathscr{V}_{\rho \not\equiv 0} &:= \{p \in \mathscr{I}^+ |\; p \textit{ is an end-point of an integral curve } \gamma \textit{ of } \ell \\ &\qquad \textit{with } \rho \not\equiv 0 \textit{ on } \gamma \;\} \subset \mathscr{V} , \\ \mathscr{U}_{\rho \not\equiv 0} &:= \{p \in \mathscr{I}^+ |\; p \textit{ is an end-point of } \underline{\textit{precisely one}} \textit{ integral curve } \gamma \textit{ of } \ell \\ &\qquad \textit{with } \rho \not\equiv 0 \textit{ on } \gamma\} \subset \mathscr{V}_{\rho \not\equiv 0} . \end{aligned}$$

Then

1. *The field $\Omega^{-2}\ell$ extends smoothly and transversally to a neighborhood of $p \in \mathscr{V}$ if and only if $p \in \mathscr{U}_{\rho \not\equiv 0}$.*
2. *The sets $\mathscr{V}_{\rho \not\equiv 0}$ and $\mathscr{U}_{\rho \not\equiv 0}$ coincide, and are open subsets of $\mathscr{I}^+$ (perhaps empty).*

Proof: Point 1: The necessity follows from Proposition 3.1, the sufficiency from Proposition B.3 below. Point 2 follows from Proposition B.3. □

An example of a set $\mathscr{V}$ which is the union of precisely one generator of $\mathscr{I}^+$ and one generator of $\mathscr{I}^-$ (and is therefore closed, without interior) is provided by the congruence of null geodesics with tangent vector $\partial_t + \partial_z$ in Minkowski space-time. Note that in this example ℓ extends to a smooth vector

field *everywhere* tangent to $\mathscr{I}$, and thus $\Omega^{-2}\ell$ extends neither to $\mathscr{I}^+$ nor to $\mathscr{I}^-$.

An example of $\mathscr{V}$ which is *not* closed is provided by the Robinson congruence in Minkowski space-time [20, Volume I, p. 59], where $\mathscr{V}$ equals $\mathscr{I}$ with one generator removed from each of $\mathscr{I}^+$ and $\mathscr{I}^-$.

Theorem B.1 has the following corollary:

COROLLARY B.2 *Let ℓ be smooth on the intersection $\mathscr{U} \cap \mathscr{M}$ of a neighborhood $\mathscr{U}$ of $\mathscr{I}^+$ with $\mathscr{M}$, and suppose that all future directed integral curves of ℓ in $\mathscr{U}$ have end points on $\mathscr{I}^+$. Then the following conditions are equivalent*

1. $\tilde{\ell} := \Omega^{-2}\ell$ *extends smoothly and transversally to* $\mathscr{I}^+$.
2. $\tilde{\rho}$ *is bounded on* $\mathscr{U}$.
3. ρ *is nowhere vanishing on* $\mathscr{U} \cap \mathscr{M}$.

PROOF: The implication *1* $\Longrightarrow$ *2* is obvious. Next, (B.15) below shows that ρ does not vanish near $\mathscr{I}^+$ under the hypothesis of point *2*, but then ρ is nowhere vanishing by (B.1) as long as the congruence remains smooth, and the implication *2* $\Longrightarrow$ *3* follows. Finally, the extendibility part of *3* $\Longrightarrow$ *1* follows from Theorem B.1; transversality follows from the construction in that Theorem. □

Before passing to the statement, and proof, of Proposition B.3, we analyse the transformation properties of the objects at hand under conformal rescalings. From the general theory of algebraically-special metrics [13, 14, 23], which has been reviewed in Section 3, there is a normalized spinor dyad (o^A, ι^A) related to the affinely-parameterized vector field ℓ by $\ell^a = o^A \overline{o}^{A'}$, and with the following restrictions on the spin-coefficients:

$$\kappa = \epsilon = \sigma = \tau = \pi = \lambda = 0.$$

In any region in which the complex expansion ρ is non-zero, the r-dependence of the non-zero spin-coefficients for vacuum, where r is an affine-parameter along ℓ so that $\ell(r) = 1$, has been explicitly found above as:

$$\rho = -(r + r^0 + i\Sigma)^{-1} \tag{B.1}$$
$$\alpha = -\alpha^0 \rho \tag{B.2}$$
$$\beta = -\beta^0 \overline{\rho} \tag{B.3}$$
$$\gamma = \gamma^0 + \frac{1}{2}\rho^2 \psi_2^0 \tag{B.4}$$
$$\mu = \mu^0 \overline{\rho} + \frac{1}{2}\rho(\overline{\rho} + \rho)\psi_2^0 \tag{B.5}$$
$$\nu = \nu^0 + \psi_3^0 \rho + \frac{1}{2}\psi_3^1 \rho^2 + \frac{1}{3}\psi_3^2 \rho^3 \tag{B.6}$$

In (B.1)-(B.6), the superscript zero indicates a function constant along ℓ and ψ_3^1, ψ_3^2 are also constant along ℓ.

For the non-vacuum case, γ, μ and ν are given instead by (3.27), (3.29) and (3.30), which will be sufficient for our conclusion below.

With the conformal rescaling $\widetilde{g} = \Omega^2 g$ we obtain ([24])

$$\widetilde{\nabla}_a \widetilde{\nabla}_b \Omega - \frac{1}{2}\Omega^{-1}\widetilde{g}_{ab}(\widetilde{g}^{ef}\partial_e\Omega\partial_f\Omega) = \Omega(-\tilde{\Phi}_{ab} + \widetilde{g}_{ab}\tilde{\Lambda}) , \tag{B.7}$$

where, following the usual NP conventions,

$$\Phi_{ab} = -\frac{1}{2}R_{ab} + \frac{1}{8}Rg_{ab} \qquad \Lambda = \frac{1}{24}R ,$$

in terms of the Ricci tensor R_{ab} and scalar curvature R, and the tilde indicates that these quantities are calculated for $\widetilde{g}$.

Now ℓ is geodesic, shear-free and affinely parameterized for g, and one readily finds that $\tilde{\ell} = \Omega^{-2}\ell$ has the same properties for $\widetilde{g}$. Suppose an affine parameter for $\tilde{\ell}$ is $\tilde{r}$, so that $\tilde{\ell}(\tilde{r}) = 1$, as well as $\ell(r) = 1$. Then $\tilde{\ell}$ is bounded in $\widetilde{\mathscr{M}}$, being a solution of the equation $\tilde{\nabla}_{\tilde{\ell}}\tilde{\ell} = 0$ with smooth data at $\Omega = \epsilon > 0$. Contract (B.7) with $\tilde{\ell}^a\tilde{\ell}^b$ to find

$$\frac{d^2\Omega}{d\tilde{r}^2} = \Omega(-\tilde{\Phi}_{ab}\tilde{\ell}^a\tilde{\ell}^b). \tag{B.8}$$

Integrate this twice along a geodesic of the congruence, fixing the origin of $\tilde{r}$ to be at $\mathscr{I}^+$ (note that $\tilde{r} \leq 0$ then), to obtain:

$$\frac{d\Omega}{d\tilde{r}} = A + \int_{\tilde{r}}^{0} \Omega(s)(\tilde{\Phi}_{ab}\tilde{\ell}^a\tilde{\ell}^b)(s)ds , \tag{B.9}$$

$$\Omega = A\tilde{r} + \int_{\tilde{r}}^{0} (\tilde{r} - s)\Omega(s)(\tilde{\Phi}_{ab}\tilde{\ell}^a\tilde{\ell}^b)(s)ds , \tag{B.10}$$

where A is a constant of integration which can be written as

$$A = \frac{d\Omega}{d\tilde{r}}\Big|_{\mathscr{I}^+} = \tilde{\ell}^a\Omega_{,a}|_{\mathscr{I}^+} .$$

(The limit is negative since Ω decreases towards $\mathscr{I}^+$).

Suppose that $p \in \mathscr{I}^+$ is an end-point of an integral curve of ℓ. Then $\tilde{\ell}$ is transverse to $\mathscr{I}^+$ at p and we conclude that A is nonzero there. We have a remaining freedom to multiply ℓ and hence also $\tilde{\ell}$ by a positive function and we may use this to set $A = -1$. Now

$$1 = \tilde{\ell}(\tilde{r}) = \Omega^{-2}\ell(\tilde{r}) = \Omega^{-2}\frac{d\tilde{r}}{dr} ,$$

from which

$$r\Omega \to 1 \quad \text{as} \quad r \to \infty , \quad \text{and} \quad \frac{d\tilde{r}}{dr} = \Omega^2 . \tag{B.11}$$

The chosen rescaling of ℓ implies the following rescalings for the null tetrad

$$\tilde{\ell}^a = \Omega^{-2}\ell^a , \quad \tilde{m}^a = \Omega^{-1}m^a , \quad \tilde{n}^a = n^a ,$$

the following for the corresponding one-forms:

$$\tilde{\ell}_a = \ell_a \; , \quad \tilde{m}_a = \Omega m_a \; , \quad \tilde{n}_a = \Omega^2 n_a \; ,$$

and the following for the spinor dyad:

$$\tilde{o}^A = \Omega^{-1} o^A \; , \quad \tilde{\iota}^A = \iota^A \; , \quad \tilde{o}_A = o_A \; , \quad \tilde{\iota}_A = \Omega \iota_A \; , \tag{B.12}$$

while the spin-coefficients change according to:

$$\begin{aligned}
\tilde{\alpha} &= \Omega^{-1}\alpha - \Omega^{-2}\,\overline{\delta}\Omega \; , && \text{(B.13)}\\
\tilde{\beta} &= \Omega^{-1}\beta \; , && \text{(B.14)}\\
\tilde{\rho} &= \Omega^{-2}(\rho - \Omega^{-1}D\Omega) \; , && \text{(B.15)}\\
\tilde{\tau} &= -\Omega^{-2}\delta\Omega \; , && \text{(B.16)}\\
\tilde{\gamma} &= \gamma - \Omega^{-1}\Delta\Omega \; , && \text{(B.17)}\\
\tilde{\pi} &= \Omega^{-2}\,\overline{\delta}\Omega \; , && \text{(B.18)}\\
\tilde{\mu} &= \mu - \Omega^{-1}\Delta\Omega \; , && \text{(B.19)}\\
\tilde{\nu} &= \Omega\nu \; , && \text{(B.20)}
\end{aligned}$$

as well as $\tilde{\epsilon} = \tilde{\kappa} = \tilde{\sigma} = \tilde{\lambda} = 0$. From (B.7)

$$\tilde{D}\tilde{\delta}\Omega - (\tilde{D}\tilde{m}^a)\partial_a\Omega = -\Omega\tilde{\Phi}_{ab}\tilde{\ell}^a\tilde{m}^b := -\tilde{\Phi}_{01}\Omega \; ,$$

or with (B.18) and the definition of $\tilde{\pi}$ and $\tilde{\delta}$:

$$\tilde{D}(\Omega^{-2}\delta\Omega) = -\tilde{\Phi}_{01} \; . \tag{B.21}$$

We are ready to prove now

Proposition B.3 *The set $\mathscr{V}_{\rho\not\equiv 0}$ is open and coincides with $\mathscr{U}_{\rho\not\equiv 0}$. Moreover the field $\Omega^{-2}\ell$ extends by continuity to a smooth vector field $\tilde{\ell}$ on $\mathscr{V}_{\rho\not\equiv 0}$.*

Proof: Consider an integral curve Γ of ℓ which has an end point on $\mathscr{I}^+$ at $p \in \mathscr{V}_{\rho\not\equiv 0}$. Then Γ can be extended to a null geodesic with tangent $\tilde{\ell}$, still denoted by Γ, which meets $\mathscr{I}^+$ transversally at p. There exists $\epsilon_0 > 0$ so that Γ meets all the level set $\{\Omega = \epsilon\}$, $0 \le \epsilon \le \epsilon_0$ transversally. Let $\mathscr{W} \subset \{\Omega = \epsilon_0\}$ be a small conditionally compact open neighborhood of $\Gamma \cap \{\Omega = \epsilon_0\}$, on which ρ is not vanishing, and to which ℓ is transverse. Let the set

$$\widetilde{\mathscr{O}} \subset \widetilde{\mathscr{M}}$$

be the union of points obtained by flowing $\mathscr{W}$ along the geodesics tangent to $\tilde{\ell}$ from $\mathscr{W}$ to $\mathscr{I}^+$. We let

$$\mathscr{O} = \widetilde{\mathscr{O}} \cap \mathscr{M}$$

denote the intersection of $\widetilde{\mathscr{O}}$ with the original space-time $\mathscr{M}$.

We start by showing that the tilded spin coefficients are uniformly bounded on $\mathscr{O}$. To see that, integrate (B.21) to find that $\Omega^{-2}\delta\Omega$ is bounded up to $\mathscr{I}^+$, and therefore, by (B.16) and (B.18), so are $\tilde{\tau}$ and $\tilde{\pi}$. From (B.1)-(B.3) and

(B.13)-(B.15), we may conclude boundedness of $\tilde\alpha$, $\tilde\beta$. For $\tilde\rho$, straightforward manipulations using (B.10) lead to the following form of (B.15):

$$\begin{aligned}\tilde\rho &= \frac{1}{\Omega(r+i\Sigma)}\Big[i\Sigma - \frac{1+r\tilde r}{\Omega} + \frac{r}{\Omega}\int_{\tilde r}^0 (\tilde r - s)\Omega(s)(\tilde\Phi_{ab}\tilde\ell^a\tilde\ell^b)(s)ds\Big] \\ &\quad -\frac{1}{\Omega}\int_{\tilde r}^0 \Omega(s)(\tilde\Phi_{ab}\tilde\ell^a\tilde\ell^b)(s)ds\ . \end{aligned} \tag{B.22}$$

Note that $\Omega r \to 1$ and $r\tilde r \to -1$ as $\tilde r$ approaches zero, and boundedness of each term in (B.22) easily follows. For $\tilde\gamma$, we return to (B.7) and contract with $\tilde\ell^a\tilde n^b$ to find

$$\tilde D(\Omega^{-1}\Delta\Omega) = |\Omega^{-2}\delta\Omega|^2 - \tilde\Phi_{11} + \tilde\Lambda\ , \tag{B.23}$$

using what we already have. Integrate this to find that $\Omega^{-1}\Delta\Omega$ is bounded at $\mathscr{I}^+$ and therefore so also is $\tilde\gamma$, from (B.4). Finally, from (B.5), B.6), (B.19) and (B.20), $\tilde\mu$ and $\tilde\nu$ are bounded.

In the non-vacuum case, we need the modified expressions (3.27), (3.29) and (3.30) for γ, μ and ν but the conclusion is the same.

Now

$$\begin{aligned}\widetilde\nabla_a\tilde\ell^b &= \tilde\ell_a((\tilde\gamma+\overline{\tilde\gamma})\tilde\ell^b - \tilde\tau\overline{\tilde m}^b - \overline{\tilde\tau}\tilde m^b) \\ &\quad -\tilde m_a((\tilde\alpha+\overline{\tilde\beta})\tilde\ell^b - \tilde\rho\overline{\tilde m}^b - \overline{\tilde\rho}\tilde m^b) \\ &\quad -\overline{\tilde m}_a((\overline{\tilde\alpha}+\tilde\beta)\tilde\ell^b - \overline{\tilde\rho}\tilde m^b - \tilde\rho\overline{\tilde m}^b)\ , \end{aligned} \tag{B.24}$$

with similar expressions for the derivatives of $\tilde n$ and $\tilde m$, and we have shown that all the covariant derivatives of the tetrad are uniformly bounded. It follows that the tetrad $\tilde\ell$, $\tilde m$, $\tilde n$, is uniformly Lipschitz, and therefore extends to a Lipschitz continuous tetrad on the $\mathscr{M}$-closure $\overline{\mathscr{O}} \supset \widetilde{\mathscr{O}}$ of $\mathscr{O}$. In particular the extended vector field $\tilde\ell$ is Lipschitz continuous. This implies that the map obtained by flowing along the geodesic with initial tangent $\tilde\ell$ from $\mathscr{I}^+$ for an affine parameter distance $\tilde r$ defines a Lipschitz continuous function of the coordinates, say v^A, on $\mathscr{I}^+$: indeed, by definition we have, in any smooth coordinate system near $\mathscr{I}^+$,

$$x^\mu(\tilde r, v^A) - x^\mu(\tilde r, v'^A) = -\int_{\tilde r}^0 \Big(\tilde\ell^\mu(x^\nu(s,v^A)) - \tilde\ell^\mu(x^\nu(s,v'^A))\Big)ds\ , \tag{B.25}$$

and the Lipschitz character of $v^A \to x^\mu(\tilde r, v^A)$ follows from the Gronwall inequality.

We now show (uniform) Lipschitz continuity of the connection coefficients. First, from (B.9)-(B.10), the functions Ω, $\Omega/\tilde r$ and $d\Omega/d\tilde r$ are now uniformly Lipschitz in the variables $(\tilde r, v^A)$ by a calculation similar to that in (B.25). Next, we want to show Lipschitz continuity of the right-hand-side of (B.22), which we rewrite in the following way, convenient for the purposes here:

$$\begin{aligned}\tilde\rho &= \frac{1}{\Omega(r+i\Sigma)}\Big[i\Sigma - \frac{1+r\tilde r}{\Omega} + r\tilde r\frac{\tilde r}{\Omega(\tilde r)}\int_{\tilde r}^0 \Big(1-\frac{s}{\tilde r}\Big)\frac{\Omega(s)}{s}\frac{s}{\tilde r}(\tilde\Phi_{ab}\tilde\ell^a\tilde\ell^b)(s)ds\Big] \\ &\quad -\frac{\tilde r}{\Omega(\tilde r)}\int_{\tilde r}^0 \frac{\Omega(s)}{s}\frac{s}{\tilde r}(\tilde\Phi_{ab}\tilde\ell^a\tilde\ell^b)(s)ds\ . \end{aligned} \tag{B.26}$$

Consider the function

$$h := r\tilde{r} + 1 \;;$$

it follows from (B.11) that h satisfies the equation

$$\tilde{r}\frac{dh}{d\tilde{r}} = h + H \;, \quad H = \Big(\frac{\tilde{r}}{\Omega}\Big)^2 - 1 \;,$$

where the function H is already known to be uniformly Lipschitz in v^A. Integration gives

$$h = C\tilde{r}\Big(1 + \int_{\tilde{r}_0}^{\tilde{r}} \frac{H(s)}{s^2} ds\Big) \;, \tag{B.27}$$

and uniform Lipschitz continuity of h — and hence also of $r\tilde{r}$ — follows by straightforward estimations. But now

$$\Omega r = \Big(\frac{\Omega}{\tilde{r}}\Big)(r\tilde{r})$$

is uniformly Lipschitz as well.

Rewriting (B.10) with $A = -1$ as

$$\frac{\Omega}{\tilde{r}} + 1 = \int_{\tilde{r}}^{0} (1 - \frac{s}{\tilde{r}})\frac{\Omega(s)}{s} s(\tilde{\Phi}_{ab}\tilde{\ell}^a\tilde{\ell}^b)(s) ds \;,$$

we find that $\Omega/\tilde{r} + 1$ is $O(\tilde{r}^2)$, with v^A-Hölder modulus of continuity also being $O(\tilde{r}^2)$. But then

$$H = \frac{\Big(1 - \frac{\Omega}{\tilde{r}}\Big)\Big(1 + \frac{\Omega}{\tilde{r}}\Big)}{\Big(\frac{\Omega}{\tilde{r}}\Big)^2}$$

is $O(\tilde{r}^2)$, with v^A-Hölder modulus of continuity $O(\tilde{r}^2)$. Rewriting (B.27) as

$$\frac{h}{\tilde{r}} = C\Big(1 + \int_{\tilde{r}_0}^{\tilde{r}} \frac{H(s)}{s^2} ds\Big) \;, \tag{B.28}$$

we conclude that $h/\tilde{r}$ is uniformly Lipschitz continuous in v^A. It follows that $h/\Omega = (h/\tilde{r})(\tilde{r}/\Omega)$ also is. From the right-hand-side of (B.26) we conclude that $\tilde{\rho}$ is uniformly Lipschitz in v^A.

To continue, integration of (B.21) shows that $\Omega^{-2}\delta\Omega$ is a Lipschitz function of v^A, which in turn justifies Lipschitz continuity of $\tilde{\tau}$ and $\tilde{\pi}$. Furthermore, the uniformly Lipschitz character of the flow of $\tilde{\ell}$ implies that all the functions such as Σ, α^0, *etc.*, are Lipschitz continuous functions of v^A, hence — by composition — Lipschitz continuous functions on $\widetilde{\mathscr{M}}$. This, together with (B.6) and (B.20) immediately shows that $\tilde{\nu}$ is Lipschitz continuous. From what has been said and from (B.1)–(B.3), (B.13)–(B.14) we conclude that $\tilde{\alpha}$ and $\tilde{\beta}$ are uniformly Lipschitz continuous. Finally, integration of the right-hand-side of (B.23) gives uniform Lipschitz continuity of $\Omega^{-1}\Delta\Omega$ and hence, in view of (B.4), (B.5), (B.17) and (B.19), that of $\tilde{\gamma}$ and $\tilde{\mu}$.

But now the right-hand-side of (B.24) is uniformly Lipschitz continuous, and hence $\widetilde{\nabla}\tilde{\ell}$ extends to a $C^{1,1}$ vector field on $\overline{\mathscr{O}}$. Similarly the remaining elements of the tetrad are $C^{1,1}$ on $\overline{\mathscr{O}}$.

The vector field $\tilde{\ell}$ is transverse to $\mathscr{I}^+$ at p by hypothesis, further $\tilde{\ell}$ is transverse to $\mathscr{W}$, and the implicit function theorem applied to the map obtained by flowing from $\mathscr{W}$ to $\mathscr{I}^+$ along $\tilde{\ell}$ provides a diffeomorphism from a neighborhood of $\Gamma\cap\mathscr{W}$ to a neighborhood of p within $\mathscr{I}^+$. This shows in particular that $\mathscr{V}_{\rho\not\equiv 0}$ contains a neighborhood of p, hence $\mathscr{V}_{\rho\not\equiv 0}$ is open. Further, every point near p is the end point of a unique element of the congruence generated by ℓ, so that $\mathscr{V}_{\rho\not\equiv 0}=\mathscr{U}_{\rho\not\equiv 0}$ near p.

One can iterate the regularity argument above as many times as the differentiability of the metric allows, obtaining each time one more degree of differentiability of $\tilde{\ell}$ which, for smooth conformal boundary extensions, proves smoothness of $\tilde{\ell}$ near p.

Since $p\in\mathscr{V}_{\rho\not\equiv 0}$ is arbitrary, Proposition B.3 follows. □

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